PARTIALLY ORDERED IDEAL PRESERVING GROUPS

By

JAN FREDERICK ANDRUS

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

June, 1958

ACKNOWLEDGMENTS

The author wishes to thank Dr. Alton T. Butson, Assistant Professor of Mathematics, for suggesting the problem treated in this dissertation and for his encouragement and guidance in working towards its solution.

He is also grateful to his wife and to George B. Morgan for their invaluable work in preparation and reproduction of the manuscript.

TABLE OF CONTENTS

I. INTRODUCTION

In recent years there has been considerable study of algebraic systems in which there is a relation of simple or partial ordering which is closely related to the algebraic operations of the system. Examples of such systems are partially ordered groups and semigroups, ordered fields and rings, vector lattices, and partially ordered linear spaces. It has been usual in studying such systems to assume a very strict connection between the operations and the ordering, namely that one or all of the operations should preserve the order relation. In a recent paper Frink (1) proposed a definition of an ideal in a partially ordered set and suggested the possibility of generalizing an ordered algebraic system by requiring the algebraic operations to preserve ideals (determined by the order relation), rather than the order relation itself. Butson (2) obtained a structure theorem for simply ordered ideal preserving groups--the suggested generalization of simply ordered groups. This dissertation is concerned with partially ordered ideal preserving groups. The principal results show how a general partially ordered ideal preserving group may be decomposed into a partially ordered group and a trivial partially ordered ideal preserving group. Consequently, many of the results concerning partially ordered groups are easily extended to partially ordered ideal preserving groups.

The multiplicative group of real numbers ordered according to magnitude is a significant example of a system which is a partially ordered ideal preserving group (actually a simply ordered ideal preserving group) but is not a partially ordered group. This system contains a maximal partially ordered subgroup--namely the positive real numbers ordered as above. Orderwise, the set of negative real numbers can be considered as the dual of the set of positive real numbers so that the structure of this particular partially ordered ideal preserving group as an ordered system is completely determined by the maximal partially ordered subgroup. Actually any lattice ordered ideal preserving group is the ordinal sum of a maximal lattice ordered subgroup and its dual. However, the structure of a general partially ordered ideal preserving group is more complicated.

II. PRELIMINARY CONCEPTS

The basic concepts of group theory may be found in (3) and those of partially ordered systems in (4). Some of the less widely-known concepts are presented in this section.

Let P be a po-set (partially ordered set). If F is a non-void subset of P, F^* will denote the set $\{x \in P \mid x \geq f$ for every $f \in F\}$ and F^+ the set $\{x \in P \mid x \leq f$ for every $f \in F\}$. The sets $(F^*)^+$ and $(F^+)^*$ will be denoted by F^{*+} and F^{+*}, respectively. It can be shown that $F \subset F^{*+}$, $F \subset F^{+*}$, $(F^{*+})^* = F^*$, and $(F^{+*})^+ = F^+$.

Definition 2.1: P is up-directed if and only if $F^* \neq \emptyset$ for every finite subset F of P. It is down-directed if and only if $F^+ \neq \emptyset$ for every finite subset F of P.

Definition 2.2: A subset J of P is an order ideal if and only if $F^* \neq \emptyset$ and $F^{*+} \subset J$ for every finite subset F of J. A subset J of P is a dual order ideal if and only if $F^+ \neq \emptyset$ and $F^{+*} \subset J$ for every finite subset F of J.

The above definition generalizes the concept of a lattice ideal. It differs from the definition suggested by Frink (1) for an ideal in a po-set in the requirement that F^* be non-void.

In the remainder of this thesis order ideals and dual order ideals will be referred to more briefly as ideals and dual ideals, respectively. No ambiguity will result since algebraic ideals will not be considered.

The following properties of P are immediate consequences of Definition 2.2.

2.3: If J is an order ideal of P, $x^{+} \subset J$ for every $x \in J$, and dually.

2.4: F^{+} is an ideal for every non-void subset F of P, and dually.

If a is a minimal element of P, $a^{*+} = \{a\}$ so that the set $\{a\}$ satisfies the conditions of Definition 2.2 for an ideal. Conversely, if $\{a\}$ is an ideal, by (2.3) it is a minimal element of P. The dual statements are also true.

2.5: A set $\{a\}$ of P is an ideal if and only if a is minimal, and dually.

Definition 2.6: An element of P which is not comparable to any other element of P is called an isolated element.

Definition 2.7: An ideal J of P is called a principal ideal if $J = x^{+}$ for some $x \in P$. A dual ideal J of P is called a dual principal ideal if $J = x^{*}$ for some $x \in P$.

Definition 2.8: A po-group P (partially ordered group) is (i) a po-set, (ii) a group, in which (iii) $a \geq b$ implies that $x + a + y \geq x + b + y$ for all $x, y \in P$. If P is a lattice satisfying (ii) and (iii), it is called an l-group (lattice ordered group). If P is a simply ordered set satisfying (ii) and (iii), it is called an o-group (simply ordered group).

The following definition generalizes the above concepts in the manner suggested by Frink. This dissertation will be primarily concerned with these generalizations.

Definition 2.9: A poip-group P (partially ordered ideal preserving group) is (i) a po-set, (ii) a group, in which (iii) if J is an ideal, x + J + y is either an ideal or a dual ideal for all $x,y \in P$. If P is a lattice which satisfies (ii) and (iii), it is called a loip-group (lattice ordered ideal preserving group). If P is a simply ordered set satisfying (ii) and (iii), it is called a soip-group (simply ordered ideal preserving group).

Let P be a poip-group. Assume that there exists a minimal element $a \in P$ and that P has a chain

$$x_1 < x_2 < x_3$$

of length two. There exists an element $t \in P$ such that $t + a = x_2$. Since $\{a\}$ is an ideal, $\{x_2\}$ is either an ideal or a dual ideal. However, this is impossible because neither x_2^+ nor x_2^* is contained in the set $\{x_2\}$.

2.10: P has a minimal element if and only if every chain of P is of length less than two.

Suppose now that P has a chain

$$x_1 < x_2$$

of length one and an isolated element x_0. By (2.4) x_2^+ is an ideal containing x_1 and x_2. There exists an element $t \in P$ such that

$t + x_2 = x_0$. Since x_0 is isolated, $\{t + x_2 = x_0, t + x_1\}^+ = \emptyset$ and $\{t + x_2 = x_0, t + x_1\}^* = \emptyset$. By Definition 2.2 $t + x_2^+$ is neither an ideal nor a dual ideal--contradicting Definition 2.9. Thus if P is of finite length, it is totally unordered--that is, it is a trivial poip-group, or it is of length one and contains no isolated elements. These statements are summarized in the following theorem:

Theorem 2.11: A non-trivial poip-group has either infinite length and no minimal elements or length one and no isolated elements.

Definition 2.12: A poip-group P is dualistic if $x + J + y$ is either an ideal or a dual ideal for every dual ideal J of P and all elements $x,y \in P$. A poip-group which is not dualistic is said to be non-dualistic.

The above classifications of poip-groups will serve as a framework for the remainder of this dissertation.

All graphical representations of poip-groups in this work are similar to the "Hasse diagrams" (4, p. 6).

Example 2.13: Any po-group P is a poip-group since, for any elements $x,y \in P$, $x + F^{*+} + y = (x + F + y)^{*+}$ whenever F and F^* are non-void subsets of P.

Example 2.14: The multiplicative groups of rational and real numbers ordered according to magnitude are soip-groups.

The additive group of rational integers and the group of integers modulo p will be represented by I and I_p, respectively.

Example 2.15: I and I_{2k} are poip-groups of length one when ordered so that $m > n$ if and only if m is even and $n = m \pm 1$. The diagram of I ordered in this manner is shown in Figure 2.16.

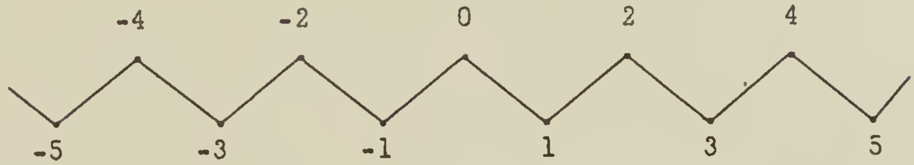


Fig. 2.16

It should be noted that there are no non-trivial po-groups of finite length.

Example 2.17: The direct sum $I_{2k} \dotplus P$ of I_{2k} ordered as in Example 2.15 and any po-group P forms a poip-group when ordered as follows: (i) if $m \neq n$, $(m, x) > (n, y)$ if and only if $m > n$, (ii) if $m = n$ and m is even, $(m, x) > (n, y)$ if and only if $x > y$, and (iii) if $m = n$ and m is odd, $(m, x) > (n, y)$ if and only if $x < y$. When $k = 1$ and P is an l-group, $I_{2k} \dotplus P$ is a loip-group under the above ordering. Figure 2.18 is the diagram for the case in which $k = 2$ and P is the additive group of rational integers in their natural order.

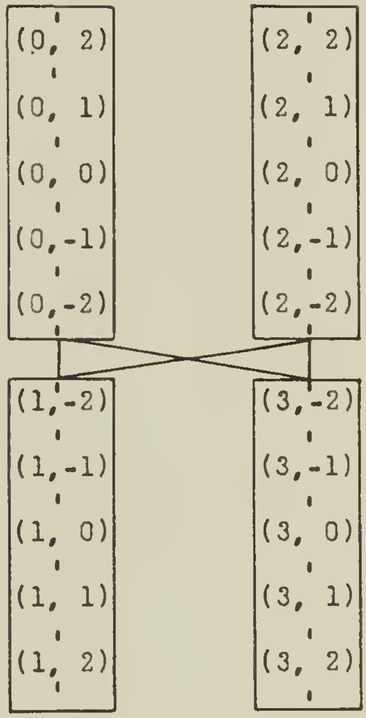


Fig. 2.18

The following is an example of a poip-group which is not the direct sum of a poip-group of finite length and a po-group.

Example 2.19: The additive group of rational integers ordered as follows is a soip-group.

$$\ldots > 2 > 0 > -2 > \ldots > -3 > -1 > 1 > 3 > \ldots$$

III. NON-DUALISTIC POIP-GROUPS

In this section we completely characterize non-dualistic poip-groups and in so doing obtain several important theorems concerning dualistic poip-groups.

Definition 3.1: Elements a and b of a po-set P are said to be weakly connected if and only if there exists a finite sequence

$$a = x_1, x_2, \ldots, x_n = b$$

of elements of P such that x_i is comparable to x_{i+1} $(i = 1, 2, \ldots, n-1)$.

Definition 3.2: A set C is said to be weakly connected if a is weakly connected to b for every pair of elements $a, b \in C$. A maximal weakly connected set is called a weak component, and C_a will denote the component containing a.

Weak connectivity is obviously an equivalence relation. It will be discussed in greater detail in Section IV but is applied in several theorems of this section.

Unless otherwise stated, all discussion will be concerned with the elements of a poip-group P. The results concerning addition of an element on the left of elements of P are true when restated in terms of addition on the right.

Lemma 3.3: If a is weakly connected to b, x + a is weakly connected to x + b.

Suppose that a is weakly connected to b. Then there exists a sequence

$$a = x_1, x_2, \ldots, x_n = b$$

such that x_i is comparable to x_{i+1}. The elements x_i and x_{i+1} are contained in either x_i^+ or x_{i+1}^+. By Definition 2.9 either $\{x + x_i, x + x_{i+1}\}^+ \neq \emptyset$ or $\{x + x_i, x + x_{i+1}\}^* \neq \emptyset$. Hence $x + x_i$ is weakly connected to $x + x_{i+1}$. It follows from the transitivity of weak connectivity that $x + a$ is weakly connected to $x + b$.

<u>Theorem 3.4</u>: The weak component C_0 is a normal subgroup of P whose cosets are the weak components of P.

Let c and c′ be elements of C_0. Then c and c′ are weakly connected to 0. By Lemma 3.3 $c + c'$ is weakly connected to $c + 0 = c$. By transitivity $c + c'$ is weakly connected to 0, implying that $c + c' \in C_0$. Again by Lemma 3.3 $-c + c = 0$ is weakly connected to $-c + 0 = -c$. Hence $-c \in C_0$. Thus C_0 is a subgroup of P.

By Lemma 3.3 $-a + C_a$ is a weakly connected set, all the elements of which are weakly connected to 0. Hence $-a + C_a \subset C_0$ and $C_a \subset a + C_0$. Likewise, $a + C_0$ is a weakly connected set, all the elements of which are weakly connected to a. Thus $C_a \supset a + C_0$ so that $C_a = a + C_0$. Similarly, $C_0 + a = C_a$.

<u>Lemma 3.5</u>: Let P be a system which is (i) a po-set, (ii) a group, in which (iii) for each weak component C of P and each pair of elements $x, y \in P$, either $x + a + y \geq x + b + y$ for all pairs $a, b \in C$

such that $a \geq b$ or $x + a + y \leq x + b + y$ for all pairs $a,b \in C$ such that $a \geq b$. Then P is a dualistic poip-group.

Let J be an ideal of P. Since $\{a, b\}^* \neq \emptyset$ for all $a,b \in J$, J is a subset of C for some weak component C of P.

Let x be any element of P and F any finite subset of J. Consider the case in which x preserves order when added to the left of elements of C (that is, $x + a \geq x + b$ for all $a,b \in C$ such that $a \geq b$). It can easily be shown that $-x$ preserves order when added to the left of elements of $x + C$. Let $x + d$ be any element of $x + F^{*+}$ and $x + e$ any element of $(x + F)^*$. Since $x + e$ is over every element of $x + F$, e is over every element of F. Hence $e \in F^*$. Therefore, $d \leq e$ so that $x + d \leq x + e$--that is, $x + d \in (x + F)^{*+}$. Hence $x + F^{*+} \subset (x + F)^{*+}$. Now let $x + g$ be any member of $(x + F)^{*+}$. It is under every element of $(x + F)^*$. If we let h be any element of F^*, it is over all members of F so that $x + h$ is over all members of $x + F$. Thus $x + h \in (x + F)^*$ and $x + F^* \subset (x + F)^*$; in fact, it can be shown that the sets are equal. Therefore, $x + g$ is under all elements of $x + F^*$--that is, g is under every element of F^* and $g \in F^{*+}$. This means $x + g \in x + F^{*+}$ so that $x + F^{*+} \supset (x + F)^{*+}$. Hence $x + F^{*+} = (x + F)^{*+}$.

It can be shown in a similar manner that if x reverses order when added to the left of elements of C, then $x + F^{*+} = (x + F)^{+*}$. Thus $x + F^{*+}$ is equal to either $(x + F)^{*+}$ or $(x + F)^{+*}$.

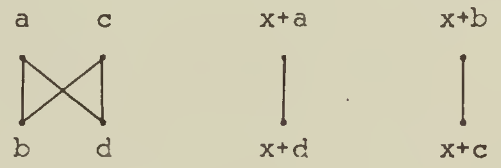
Suppose $x + F^{*+} = (x + F)^{*+}$. Since $x + F^{*+} \subset x + J$, $(x + F)^{*+} \subset x + J$. Now any finite subset of $x + J$ can be expressed in the form $x + F$ where F is a finite subset of J. Therefore $x + J$ is an ideal.

On the other hand, it can be shown that if $x + F^{*+} = (x + F)^{+*}$, $x + J$ is a dual ideal.

Hence for any ideal J of P and $x \in P$, $x + J$ is either an ideal or a dual ideal. Dually, $x + J$ is either an ideal or a dual ideal if J is a dual ideal of P. Clearly $x + J + y$ is an ideal or a dual ideal for any ideal or dual ideal J of P and any pair $x, y \in P$. Thus P is a dualistic poip-group.

The following four lemmas on poip-groups of finite dimension will be used in the proof of Theorem 3.10, which completely characterizes non-dualistic poip-groups.

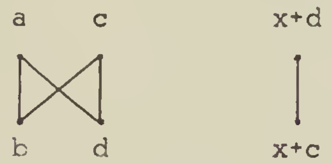
<u>Lemma 3.6</u>: In a poip-group P of length one the figures



cannot exist simultaneously.

Assume that the figures do exist in P. By (2.4) $\{a, c\}^{+}$ is an ideal containing b and d. The elements $x + b$ and $x + d$ are in $x + \{a, c\}^{+}$. Therefore $x + \{a, c\}^{+}$ is not an ideal since $x + c \in (x + b)^{+}$ but $c \notin \{a, c\}^{+}$, and it is not a dual ideal since $x + a \in (x + d)^{*}$ but $a \notin \{a, c\}^{+}$. This is a contradiction since $x + a^{+}$ must be either an ideal or a dual ideal.

<u>Lemma 3.7</u>: In a poip-group P of length one the figures



cannot exist simultaneously if $x + a$ and $x + b$ are maximal.

Assume that the figures do exist simultaneously and that $x + a$ and $x + b$ are maximal. The ideal $(x + d)^{+}$ contains $x + c$ and $x + d$ but contains neither $x + a$ nor $x + b$. Hence $-x + (x + d)^{+}$ contains c and d but does not contain a and b. By (2.3) $-x + (x + d)^{+}$ is neither an ideal nor a dual ideal. Since it must be one or the other, this is a contradiction.

<u>Lemma 3.8</u>: In a poip-group P of length one $x + a$ and $x + b$ are not both maximal if $a > b$.

Assume that $a > b$ and that $x + a$ and $x + b$ are maximal. Since $x + a^{+}$ is either an ideal or a dual ideal and contains $x + a$ and $x + b$, there exists an element $x + c \in \{x + a, x + b\}^{+}$. Obviously $(x + a)^{+}$ and $(x + b)^{+}$ are ideals containing $x + c$.

Since $-x + (x + a)^{+}$ is either an ideal or a dual ideal, either $\{a, c\}^{+} \neq \emptyset$ or $\{a, c\}^{*} \neq \emptyset$. Also $-x + (x + b)^{+}$ is either an ideal or a dual ideal, so that either $\{b, c\}^{+} \neq \emptyset$ or $\{b, c\}^{*} \neq \emptyset$. Clearly this is impossible unless either $c > b$ or $c < a$.

<u>Case i</u>: Suppose $c > b$. The set $-x + (x + a)^{+}$ contains a and c but not b since $x + b$ is maximal. By (2.3) and the fact that $b \in a^{+}$, $-x + (x + a)^{+}$ is a dual ideal. If $\{b\} = \{a, c\}^{+}$, then

$b \in \{a, c\}^{+*}$. By the definition of a dual ideal $\{b\} = \{a, c\}^{+}$ implies $b \in -x + (x + a)^{+}$. Hence $\{b\} \neq \{a, c\}^{+}$, so that there must be an element $d \in \{a, c\}^{+}$ distinct from b. The set $x + a^{+}$ contains $x + a$ and $x + d$, so that either $\{x + a, x + d\}^{+} \neq \emptyset$ or $\{x + a, x + d\}^{*} \neq \emptyset$. Since $x + c^{+}$ contains $x + c$ and $x + d$, either $\{x + c, x + d\}^{+} \neq \emptyset$ or $\{x + c, x + d\}^{*} \neq \emptyset$. The above conditions can occur only if $x + d < x + a$ or $x + d > x + c$. However, these relations contradict Lemmas 3.6 and 3.7, respectively.

Case ii: Suppose $c < a$. The set $-x + (x + b)^{+}$ contains b and c but not a since $x + a$ is maximal. By an argument similar to that in Case i, there exists an element d such that $d > b$, $d > c$, and $d \neq a$. Since $x + d^{+}$ contains $x + b$, $x + c$, and $x + d$, either $\{x + b, x + c, x + d\}^{+} \neq \emptyset$ or $\{x + b, x + c, x + d\}^{*} \neq \emptyset$. Hence either $x + d < x + b$ or $x + d > x + c$. As before, these conditions contradict Lemmas 3.6 and 3.7, respectively. This establishes the lemma.

Lemma 3.9: If P is a poip-group of length one in which $a \geq b$ implies $x + a + y$ and $x + b + y$ are comparable for all $x, y \in P$, then P is dualistic.

Let c and c′ be maximal elements in a weak component C of P. Then there exists a sequence

$$c = x_1, x_2, \ldots, x_n = c'$$

of distinct elements of C such that x_i is comparable to x_{i+1}. We note that x_i is maximal if and only if i is odd and in particular that n is odd. The sequence

$$x + c = x + x_1, x + x_2, \ldots, x + x_n = x + c'$$

is also a sequence of distinct elements such that $x + x_i$ is comparable to $x + x_{i+1}$. If $x + c$ is maximal (minimal), then $x + x_i$ is maximal if and only if i is odd (even). Since n is odd, $x + c' = x + x_n$ is maximal if and only if $x + c$ is maximal.

Thus if c' is some maximal element of C and if $x + c'$ is maximal (minimal), then $x + c$ is maximal (minimal) for all maximal elements $c \in C$. Since every element of P is either maximal or minimal, x preserves (reverses) order when added on the left of the elements of C. It follows that P satisfies conditions i-iii of Lemma 3.5 and that it is dualistic.

Theorem 3.10: Any non-dualistic poip-group is (i) a po-set of length one, in which (ii) the weak components are the cosets of C_0, (iii) every weak component is a principal ideal, and (iv) C_0 contains more than two distinct elements. Conversely, any group P satisfying conditions i-iv for some normal subgroup C_0 is a non-dualistic poip-group.

The latter statement will be proved first. Let P be a group satisfying conditions i-iv for some normal subgroup C_0 of P, and let J be an ideal of P. By Definition 2.2 J is contained in some weak component C, which by condition iii is a principal ideal, say, c^+. If J is an ideal $\{j\}$ consisting of a single element, then $x + J + y = \{x + j + y\}$ is, according to (i), either an ideal or a dual ideal. If J contains two distinct elements j and j′, by Definition 2.9 it contains $\{j, j'\}^{*+}$. Thus $c \in J$ so that $J \supset c^+ = C$. Hence $J = C$. Since $x + J + y = x + C + y = x + C_c + y = C_{x+c+y}$, $x + J + y$ is a principal ideal. Therefore, P is a poip-group.

In order to show that P is non-dualistic, consider a weak component c^+ and distinct elements c_1 and c_2, which are properly under c. That such elements exist follows from condition iv. Now there is an x such that $x + c_2 = c$. Since c, c_1, and c_2 are distinct, $x + c \neq c$ and $x + c_1 \neq c$. Thus $x + c$ and $x + c_1$ are minimal elements such that $\{x + c, x + c_1\}^* = \{x + c_2\} = \{c\}$. Hence $\{c\} \in \{x + c, x + c_1\}^{*+}$ but $\{c\} \notin \{x + c, x + c_1\}^* = x + c_1^*$. This implies that $x + c_1^*$ is not an ideal. It also is not a dual ideal since $\{x + c, x + c_1\}^+ = \emptyset$. Thus P is a non-dualistic poip-group.

Now let P be a non-dualistic poip-group of finite dimension. By the contrapositive of Lemma 3.9 there exist elements $a, b, x \in P$ such that $a > b$ but $x + a$ and $x + b$ are not comparable. By Lemma 3.8 and the fact that $\{x + a, x + b\} \subset x + a^+$, there exists an element $x + c$ properly over $x + a$ and $x + b$.

The elements a, b, and c are members of $-x + (x + c)^+$, so that either $\{a, b, c\}^+ \neq \emptyset$ or $\{a, b, c\}^* \neq \emptyset$. This implies either $c > b$ or $c < a$. If $c > b$, then a and c are maximal and $x + c > x + a$, which contradicts Lemma 3.8. So we must have $c < a$.

The set $x + a$ is not a dual ideal since it contains two minimal elements, $x + a$ and $x + b$. Thus it is an ideal. It cannot contain a maximal element distinct from $x + c$ or a minimal element not under $x + c$. Hence $x + a^+ \subset (x + c)^+$. Clearly $x + a^+ \supset (x + c)^+$, so that $x + a^+ = (x + c)^+$.

Suppose there exists an element e such that $e > c$. By Lemma 3.8, $x + e$ is minimal. Since $\{x + c, x + e\} \subset x + e^+$, $x + c > x + e$. Thus $x + e \in (x + c)^+ = x + a^+$, so that $e \leq a$. Since e is maximal, $e = a$. This shows that $e^* = \{a, e\}$.

Now assume that there is an element f such that $f > b$. There exists y such that $y + b = c$. So $\{y + a, y + b = c, y + c\} \subset y + a^+$, which implies that either $\{y + a, y + b = c, y + c\}^* \neq \emptyset$ or $\{y + a, y + b = c, y + c\} \neq \emptyset$. Now c is minimal and $c^* = \{a, c\}$. This means that $\{y + a, y + b = c, y + c\}^+ = \emptyset$ so that $\{a\} = \{y + a, y + b = c, y + c\}^*$. Hence $a \in \{y + a, y + b = c, y + c\}^{*+}$, meaning $a \in y + a^+$. It follows from Definition 2.9 and the equality $\{y + a, y + b = c, y + c\}^+ = \emptyset$ that $y + a^+$ is an ideal and $a^+ \subset y + a^+$.

Since $y + f$ and $y + b$ are in $y + f^+$, $\{y + f, y + b = c\}^* \neq \emptyset$ or $\{y + f, y + b = c\}^+ \neq \emptyset$. Now $c^* = \{a, c\}$, which means that $y + f \in a^+$. Therefore $y + f \in y + a^+$ and $f \in a^+$. Because $f = a$, f is maximal, which shows that $b^* = \{a, b\}$.

Replacing b by d and repeating the above argument, it can be shown that $d^* = \{a, d\}$ for any d such that $d < a$. Therefore the particular component C_a is the set a^+.

We will next show that every weak component of P is a principal ideal.

Consider a weak component $\hat{C}$ and assume that it is not a principal ideal. Then there are distinct elements $c_0, c_1, c_2 \in C$

such that $c_1 > c_0$ and $c_2 > c_0$. There exists an element z such that $z + c_1 = a$. By Definition 2.9 either $\{z + c_0, z + c_i\}^* \neq \emptyset$ or $\{z + c_0, z + c_i\}^+ \neq \emptyset$ (i = 1, 2). So $\{z + c_0, z + c_1, z + c_2\} \subset a^+$. Consequently, we have $z + c_0 < z + c_1 = a$ and $z + c_2 < z + c_1 = a$. Since $\{z + c_0, z + c_2\} \subset z + c_2^+$ and $z + c_1 \notin z + c_2^+$, it follows that $z + c_2^+$ is not a dual ideal. Then $z + c_1 = a \in \{z + c_0, z + c_2\}^{*+} \subset z + c_2^+$ so that $c_1 < c_2$. This is a contradiction--implying that C is indeed a principal ideal.

So P satisfies condition iii. By Theorem 3.4 it satisfies condition ii, and it obviously fulfills conditions i and iv.

The proof of this theorem would be complete if it were known that every non-dualistic poip-group has finite dimension. This result, along with a more useful definition of a dualistic poip-group, will be obtained in the remainder of this section. However, before proceeding we give an example of a non-dualistic poip-group.

<u>Example 3.11</u>: Order the additive group of integers by placing 0 over 2n and 1 over 2n + 1 for all integers n. The resulting system is a non-dualistic poip-group with two weak components one composed of the even integers and the other composed of the odd integers.

<u>Lemma 3.12</u>: If $a > b$ in a poip-group P, then $x + a \not\leq x + b$ when $x + b^+$ is an ideal, and $x + a \not\geq x + b$ when $x + b^+$ is a dual ideal.

Assume $x + b^+$ is an ideal and $x + a \leq x + b$. Then $x + a \in x + b^+$ which implies $a \leq b$. Hence $x + a \not\leq x + b$.

Suppose $x + b^+$ is a dual ideal and $x + a \geq x + b$. Then $x + a \in x + b^+$ which means $a \leq b$. Hence $x + a \not\geq x + b$.

Lemma 3.13: If $x + b^+$ is an ideal of a poip-group P and b is not maximal, then $x + b^+ = (x + b)^+$.

Since $x + b \in x + b^+$, obviously $(x + b)^+ \subset x + b^+$. So $-x + (x + b)^+ \subset b^+$. If $-x + (x + b)^+$ were a dual ideal, we would have $b^* \subset -x + (x + b)^+ \subset b^+$. But $b^* \subset b^+$ is possible only if b is a maximal element. Thus $-x + (x + b)^+$ is an ideal. Since $b \in -x + (x + b)^+$, $b^+ \subset -x + (x + b)^+$ and $x + b^+ \subset (x + b)^+$. Therefore, $x + b^+ = (x + b)^+$.

Lemma 3.14: Let a be a non-maximal element of a poip-group P. If $a > b$, $-x + (x + a)^+$ is a dual ideal, and $x + b^+$ is an ideal, then $x + a > x + b$.

Since a is not maximal, there is an element $c > a$ in P. The dual ideal $-x + (x + a)^+$ contains a and hence c. So $x + c \in (x + a)^+$ which gives $x + c < x + a$. By the contrapositive of Lemma 3.12, $x + a^+$ is not an ideal. Therefore it must be a dual ideal. Now $\{x + a, x + b\} \subset x + a^+$, so that $x + a$ and $x + b$ have a lower bound, say, $x + g$. By Lemma 3.13 $x + b^+ = (x + b)^+$, implying that $x + g \in x + b^+$ and $b \in g^*$. Also $x + g \in (x + a)^+$ which means $g \in -x + (x + a)^+$ and $g^* \subset -x + (x + a)^+$. Hence $b \in -x + (x + a)^+$, implying that $x + b \in (x + a)^+$ which gives $x + b < x + a$.

Lemma 3.15: If $a > b$ in an infinite dimensional poip-group P, and a is not maximal, then $x + a$ and $x + b$ are comparable.

Obviously $a \in -x + (x + a)^+$. If $-x + (x + a)^+$ is an ideal, it contains a^+ and hence b. Thus if $-x + (x + a)^+$ is an ideal, $x + b \in (x + a)^+$ and $x + b < x + a$. So assume $-x + (x + a)^+$ is a dual ideal.

Clearly $b \in -x + (x + b)^+$. If $-x + (x + b)^+$ is a dual ideal, it contains b^* and thus a. Hence if $-x + (x + b)^+$ is a dual ideal, $x + a \in (x + b)^+$ and $x + a < x + b$. So assume that $-x + (x + b)^+$ is an ideal.

<u>Case i</u>: Suppose $x + b$ is not maximal. By Lemma 3.13 $-x + (x + b)^+ = b^+$ (replace x by $-x$ and b by $x + b$). Hence $(x + b)^+ = x + b^+$ so that $x + b^+$ is an ideal. It follows from Lemma 3.14 that $x + a > x + b$.

<u>Case ii</u>: Suppose $x + b$ is maximal. If $x + b^+$ is an ideal, then by Lemma 3.14 $x + a$ and $x + b$ are comparable. Therefore, assume $x + b^+$ is a dual ideal. Since P is of infinite length, it follows from (2.10) that it has no minimal element. So there must exist an element $d < b$. Now $-x + (x + b)^+$ is an ideal containing b and hence d. Thus $x + d \in (x + b)^+$ and $x + d < x + b$. Now $a > d$ where a and $x + d$ are not maximal. Replacing b by d in the proof of Case i, it follows that $x + a$ is comparable to $x + d$. If $x + a < x + d$, then $x + a < x + b$. So assume that $x + a > x + d$. Since $d \in b^+$, $x + d \in x + b^+$. Then $(x + d)^* \subset x + b^+ \subset (x + b)^+$. This gives $x + a \in (x + b)^+$ and $x + a < x + b$, completing the proof.

In Section II it was shown that an infinite dimensional poip-group P contains no minimal elements. If we can show that such a poip-group contains no maximal elements either, we shall be able to conclude from the above lemma that comparability is preserved under addition in an infinite dimensional poip-group.

Suppose a is a maximal element of a poip-group P of infinite length. Since P has no minimal elements, there exist a chain

$$a > e > f > g$$

and elements $t, b, c \in P$ such that $t + a = f$, $t + b = e$, and $t + c = g$. The above chain now becomes

$$a > t + b > t + a > t + c.$$

Since $t + b > t + a$, it follows from Lemma 3.15 that a and b are comparable. Moreover, $a > b$ because a is maximal.

By Lemma 3.12, $-t + (t + a)^+$ is a dual ideal (replace x by $-t$, a by $t + b$, and b by $t + a$). Now $t + c \in (t + a)^+$, so that $c \in -t + (t + a)^+$. Thus $c^* \subset -t + (t + a)^+$ and $t + c^* \subset (t + a)^+$. If $b \in c^*$, then $t + b \in (t + a)^+$, which is impossible since $t + b > t + a$. Hence $b \notin c^*$. However, by Lemma 3.15 b and c are comparable, so $b < c$. The elements a and c are also comparable because $t + a > t + c$ and $t + a$ is not maximal. Since a is maximal, $a > c$. From Lemma 3.12 and the relations $t + a > t + c$ and $a > c$, it follows that $-t + (t + c)^+$ is an ideal. By Lemma 3.13 $-t + (t + c)^+ = c^+$. Since $b \in c^+$, $t + b \in (t + c)^+$ and $t + b < t + c$, a contradiction. Hence P cannot contain a maximal element, and we can now state the following theorems.

<u>Theorem 3.16</u>: An infinite dimensional poip-group contains no maximal or minimal elements.

<u>Theorem 3.17</u>: Comparability is preserved under addition in an infinite dimensional poip-group--that is, if a and b are comparable, then $x + a + y$ and $x + b + y$ are comparable for all $x, y \in P$.

<u>Lemma 3.18</u>: If $a > b$ in an infinite dimensional poip-group, then (i) $x + a^+$ is an ideal if and only if $x + a > x + b$, (ii) $x + a^+$ is a dual ideal if and only if $x + a < x + b$, (iii) $x + b^+$ is an ideal if and only if $x + a > x + b$, and (iv) $x + b^+$ is a dual ideal if and only if $x + a < x + b$.

Conditions iii and iv follow from Theorem 3.17 and Lemma 3.12.

Suppose $x + a^+$ is an ideal. By Lemma 3.13 $x + a^+ = (x + a)^+$. Since $b \in a$, $x + b \in (x + a)^+$ and $x + b < x + a$.

Now suppose that $x + a^+$ is a dual ideal and $x + a > x + b$. By Lemma 3.12 $x + b^+$ is an ideal. It follows from Theorem 3.16 that there is an element $c > a$. Again by Lemma 3.12 $x + c < x + a$ and $x + c > x + b$. Hence $x + a > x + c > x + b$. The inequalities $x + a > x + c$, $c > a$, and Lemma 3.12 imply that $-x + (x + c)^+$ is a dual ideal. Now $x + b \in (x + c)^+$, so that $b \in -x + (x + c)^+$. Then $b^* \subset -x + (x + c)^+$, $a \in -x + (x + c)^+$, and $x + a \in (x + c)^+$. This implies $x + a < x + c$, a contradiction.

Thus if $x + a^+$ is an ideal, $x + a > x + b$, and if $x + a^+$ is a dual ideal, $x + a < x + b$. These statements and their contrapositives give conditions i and ii.

An immediate consequence of this lemma is the following result.

Lemma 3.19: Let a and b be comparable elements of an infinite dimensional poip-group. Then $x + a^+$ is an ideal if and only if $x + b^+$ is an ideal, and $x + a^+$ is a dual ideal if and only if $x + b^+$ is a dual ideal.

Lemma 3.20: An infinite dimensional poip-group P is (i) a po-set, (ii) a group, in which (iii) for each weak component C of P and each pair of elements $x, y \in P$ either $x + a + y \geq x + b + y$ for all pairs $a, b \in C$ such that $a \geq b$ or $x + a + y \leq x + b + y$ for all pairs $a, b \in C$ such that $a \geq b$. Conversely, any system P satisfying conditions i-iii is a dualistic poip-group.

Let x be any element and C any weak component of P. First suppose that $x + c_0^+$ is an ideal for some $c_0 \in C$. For any other $c \in C$, there exists a sequence

$$c_0 = x_1, x_2, \ldots, x_n = c$$

such that x_i and x_{i+1} are comparable. If $x + x_i^+$ is an ideal, by Lemma 3.19 $x + x_{i+1}^+$ must be an ideal. Hence $x + c^+$ is an ideal. Similarly, if $x + c_0^+$ is a dual ideal for some $c_0 \in C$, $x + c^+$ is a dual ideal for all $c \in C$. That P satisfies condition iii now follows from Lemma 3.18 and the extension of the above argument to the case in which an arbitrary element y is added on the right of elements of C.

This lemma and Lemma 3.5 enable us to conclude the following result which we noted before was necessary to complete the proof of Theorem 3.10.

<u>Theorem 3.21</u>: Every infinite dimensional poip-group is dualistic.

<u>Theorem 3.22</u>: A dualistic poip-group P is (i) a po-set, (ii) a group, in which (iii) for each weak component C of P and each pair of elements $x,y \in P$ either $x + a + y \geq x + b + y$ for all pairs $a,b \in C$ such that $a \geq b$ or $x + a + y \leq x + b + y$ for all pairs $a,b \in C$ such that $a \geq b$. Conversely, any system P satisfying conditions i-iii is a dualistic poip-group.

The latter statement of the theorem is merely Lemma 3.5.

A dualistic poip-group P obviously satisfies conditions i and ii. By Lemma 3.20, if it is of infinite dimension, it also satisfies condition iii. Hence it remains to be shown that any dualistic poip-group of dimension one satisfies condition iii.

So assume that P has length one. Consider elements $a,b \in P$ such that $a > b$ and suppose $x + a$ and $x + b$ are not comparable for some $x \in P$. Now $\{x + a, x + b\} \subset x + a^{+}$, so that either $\{x + a, x + b\}^{*} \neq \emptyset$ or $\{x + a, x + b\}^{+} \neq \emptyset$. Since $x + a$ and $x + b$ are not comparable, there exists an element which is either properly over them or is properly under them. Hence $x + a$ and $x + b$ are either both maximal or both minimal. By Lemma 3.8 and its dual, this is impossible. Therefore $x + a$ and $x + b$ are comparable.

In the proof of Lemma 3.9 it was shown that if P is a poip-group of length one in which $a \geq b$ implies $x + a + y$ and $x + b + y$ are comparable for all $x,y \in P$, then P satisfies condition iii. This completes the proof.

Since non-dualistic poip-groups were described completely in Theorem 3.10, only dualistic ones will be considered in the sequel. Also, the above theorem provides us with a more useful definition of a dualistic poip-group, which will be employed in the following sections.

IV. CONNECTIVITY

The immediately preceding theorem suggests the following definitions, which will lead to an even finer partitioning of a dualistic poip-group than that determined by the weak components.

Definition 4.1: An element x is said to be order preserving on the left relative to the weakly connected set C if $x + a \geq x + b$ for all pairs $a > b$ in C. It is said to be order reversing on the left relative to C if $x + a < x + b$ for all pairs $a > b$ in C.

By Theorem 3.22 any element of P is either order preserving or order reversing on the left relative to any given weak component.

Henceforth there will be little occasion to consider ideals. However, it should be noted that if J is an ideal contained in the weak component C and if x is order preserving (reversing) on the left relative to C, then x + J is an ideal (dual ideal). Also, if F is a non-void subset of the weak component C, then $x + F^{+} = (x + F)^{+}$ or $x + F^{+} = (x + F)^{*}$ depending on whether x is order preserving or order reversing on the left relative to C. These results are contained in the proof of Lemma 3.5.

Definition 4.2: An element x is of order preserving type 1 relative to the weakly connected set C if it is order preserving on the left and on the right relative to C. It is of order preserving type 2 relative to C if it is order preserving on the left and order reversing on the right relative to C. It is of order preserving *type 3 relative to C* if it is order reversing on the left and order preserving on the right relative to C. It is of *order preserving type 4 relative to C* if it is order reversing on the left and on the right relative to C. An element of order preserving type 1 (type 2) relative to C_a and an element of order preserving type 4 (type 3) relative to C_a are said to be of opposite order preserving types relative to C_a. The set of all elements of C_x which are of order preserving type i relative to C_a will be denoted by $C_x^i(C_a)$ (i = 1, 2, 3, 4).

We note that $C_x^i(C_a)$ and $C_x^j(C_a)$, $i \neq j$, are disjoint.

Any weakly connected set partitions P into from one to four classes of element types. If it is commutative, all elements of P are either of order preserving type 1 or order preserving type 4 relative to any given weakly connected set.

Example 4.3: Let P be generated as a group by a and x under the conditions $x + x = 0$ and $x + a = -a + x$. When it is ordered as shown in Figure 4.4, P is a non-commutative dualistic poip-group. It can be verified that a is in $C_0^1(C_0)$ and $C_0^1(C_x)$ and that x is a member of $C_x^2(C_0)$ and $C_x^2(C_x)$.

$2a$	$x + 2a = -2a + x$
a	$x + a = -a + x$
0	x
$-a$	$x - a = a + x$
$-2a$	$x - 2a = 2a + x$

Fig. 4.4

Lemma 4.5: If x is order preserving (reversing) on the left relative to C_a, -x is order preserving (reversing) on the left relative to C_{x+a}.

Let a_1 and a_2 be elements of C_a such that $a_1 \geq a_2$ and suppose that x is order preserving on the left relative to C_a, so that $x + a_1 \geq x + a_2$. The element -x is order preserving on the left relative to C_{x+a} because $-x + (x + a_1) = a_1 \geq a_2 = -x + (x + a_2)$. The remaining case can be proved in a similar manner.

Theorem 4.6: $C_0 = C_0^1(C_0) \cup C_0^4(C_0)$.

For any element $y \in C_0$ there exists a sequence

$$0 = x_1, x_2, \ldots, x_n = y$$

of distinct elements of C_0 such that x_i is comparable to x_{i+1}. Clearly $0 \in C_0^1(C_0)$, so to complete the proof it is necessary to show only that $x_i \in C_0^1(C_0) \cup C_0^4(C_0)$ implies $x_{i+1} \in C_0^1(C_0) \cup C_0^4(C_0)$.

Assume $x_i \in C_0^1(C_0)$ and $x_{i+1} \in C_0^2(C_0)$. By Lemma 4.5 $-x_i \in C_0^1(C_0)$ and $-x_{i+1} \in C_0^2(C_0)$.

Case i: Suppose $x_i > x_{i+1}$. It follows that $x_i - x_{i+1} < 0$. However, it is also true that $0 > x_{i+1} - x_i$ whence $-x_{i+1} > -x_i$ and $x_i - x_{i+1} > 0$, a contradiction.

Case ii: Suppose $x_i < x_{i+1}$. Obviously $x_i - x_{i+1} > 0$. But $0 < x_{i+1} - x_i$, so that $-x_{i+1} < -x_i$ and $x_i - x_{i+1} < 0$, which is impossible.

This proves that $x_i \in C_0^1(C_0)$ implies $x_{i+1} \notin C_0^2(C_0)$. Continuing with this procedure it can be shown that $x_i \in C_0^1(C_0) \cup C_0^4(C_0)$ implies that $x_{i+1} \notin C_0^2(C_0) \cup C_0^3(C_0)$ and, therefore, that $x_{i+1} \in C_0^1(C_0) \cup C_0^4(C_0)$. The theorem follows.

<u>Theorem 4.7</u>: For all $e \in P$, $C_0^1(C_0) = C_0^1(C_e)$ and $C_0^4(C_0) = C_0^4(C_e)$.

Let $e_1, e_2 \in C_e$ be such that $e_1 > e_2$ and consider an element $s \in C_0^1(C_0)$.

<u>Case i</u>: Suppose e_1 is order preserving on the right relative to C_0. By Lemma 4.5 $-e_1$ is order preserving on the right relative to C_e so that $0 > e_2 - e_1$. Therefore, $s > s + (e_2 - e_1) = (s + e_2) -e_1$ and $s + e_1 > s + e_2$.

<u>Case ii</u>: Suppose e_1 is order reversing on the right relative to C_0. By Lemma 4.5 $0 < e_2 - e_1$. Thus $s < s + (e_2 - e_1) = (s + e_2) - e_1$ and $s + e_1 > x + e_2$.

These arguments show that s is order preserving on the left relative to C_e. By left-right symmetry, $s \in C_0^1(C_e)$.

Similarly, $s \in C_0^4(C_0)$ implies $s \in C_0^4(C_e)$, enabling us to conclude that $C_0^1(C_0) \subset C_0^1(C_e)$ and $C_0^4(C_0) \subset C_0^4(C_e)$. However, by Theorem 4.6 $C_0 = C_0^1(C_0) \cup C_0^4(C_0)$. Since $C_0^1(C_e) \cap C_0^4(C_e) = \emptyset$, $C_0^1(C_0) = C_0^1(C_e)$ and $C_0^4(C_0) = C_0^4(C_e)$.

As a result of this theorem, no ambiguity will arise if $C_0^1(C_e)$ and $C_0^4(C_e)$ are denoted by C_0^1 and C_0^4, respectively.

<u>Theorem 4.8</u>: The set C_0^1 is a normal subgroup of P, and C_0^4 either is empty or is one of its cosets. For any $e \in P$,

$C_e = (e + C_0^1) \cup (e + C_0^4)$. Furthermore, for any $f \in P$, $e + C_0^1 = C_e^i(C_f)$ and $e + C_0^4 = C_e^j(C_f)$ where i is the order preserving type of e relative to C_f and j is the opposite order preserving type relative to C_f.

Obviously $C_e = e + C_0 = e + (C_0^1 \cup C_0^4) = (e + C_0^1) \cup (e + C_0^4)$ and $C_e = C_0 + e = (C_0^1 \cup C_0^4) + e = (C_0^1 + e) \cup (C_0^4 + e)$. Now let f_1 and f_2 be elements of C_f such that $f_1 > f_2$ and let $e + s$ be any member of $e + C_0^1$. Since $s \in C_0^1 = C_0^1(C_f)$, $s + f_1 > s + f_2$. Clearly $s + f_1 \in C_0 + f_1 = C_f$, so that $(e + s) + f_1 > (e + s) + f_2$ if e is order preserving on the left relative to C_f, and $(e + s) + f_1 < (e + s) + f_2$ if e is order reversing on the left relative to C_f. In other words, $e + s$ is order preserving (reversing) on the left relative to C_f when e is order preserving (reversing) on the left relative to C_f. Similarly, it is order preserving (reversing) on the right relative to C_f when e is order preserving (reversing) on the right relative to C_f. This shows that if i is the order preserving type of e relative to C_f, then $e + C_0^1 \subset C_e^i(C_f)$. By left-right symmetry, $C_0^1 + e \subset C_e^i(C_f)$.

It may be proved in much the same manner that if i is the order preserving type of e relative to C_f, then $e + C_0^4 \subset C_e^j(C_f)$ and $C_0^4 + e \subset C_e^j(C_f)$ where i and j are of opposite order preserving types relative to C_f. Since $(e + C_0^1) \cup (e + C_0^4) = C_e$ and $C_e^i(C_f) \cap C_e^j(C_f) = \emptyset$, it is true that $e + C_0^1 = C_e^i(C_f)$ and $e + C_0^4 = C_e^j(C_f)$.

Likewise $C_0^1 + e = C_e^i(C_f)$ and $C_0^4 + e = C_e^j(C_f)$. It follows that $e + C_0^1 = C_0^1 + e$ and $e + C_0^4 = C_0^4 + e$.

If s and t are any members of C_0^1, then $s + t \in s + C_0^1 = C_0^1(C_0) = C_0^1$. By Lemma 4.5 $s \in C_0^1$ implies $-s \in C_0^1$. Hence C_0^1 is a normal subgroup of P.

If C_0^4 is not empty, there exists an $r \in C_0^4$. Then $r + C_0^1 = r + C_0^1(C_0) = C_0^4(C_0)$, so that C_0^4 is a coset of C_0^1. This completes the proof.

Corollary 4.9: For any weak component C and any $f \in P$, either $C = C^1(C_f) \cup C^4(C_f)$ or $C = C^2(C_f) \cup C^3(C_f)$.

The decomposition of P into the cosets of C_0^1 yields little information concerning the structure of P, but it motivates the investigation of another decomposition determined by the maximal weakly connected set in C_0^1.

Definition 4.10: The element a is strongly connected to b if and only if there exists a sequence

$$a = x_1, x_2, \ldots, x_n = b$$

such that $x_i \in C_a^j(C_0)$ for some j and $i = 1, 2, \ldots, n$ and such that x_i is comparable to x_{i+1} for $i = 1, 2, \ldots, n - 1$.

Obviously if two elements are strongly connected, they are weakly connected. Note that strong connectivity is an equivalence relation on a poip-group but does not necessarily have meaning for a po-set per se.

Definition 4.11: A set S is strongly connected if and only if every two elements of S are strongly connected. A maximal strongly connected set is called a strong component of P, and S_a will denote the strong component containing a.

Any strongly connected set consists of elements of only one order preserving type relative to any weakly connected set. Obviously $S_0 \subset C_0^1 \subset C_0$.

Lemma 4.12: If a is strongly connected to b, then x + a is strongly connected to x + b.

There exists a sequence

$$a = x_1, x_2, \ldots, x_n = b$$

of elements of S_a such that x_i is comparable to x_{i+1}. Now $S_a \subset C_a^j(C_0)$ for some j so that $x + S_a \subset x + C_a^j(C_0) = C_{x+a}^k(C_0)$ for some k. In particular, the members of the sequence

$$x + a = x + x_1, x + x_2, \ldots, x + x_n = x + b$$

are contained in $C_{x+a}^k(C_0)$. Furthermore, by Theorem 3.22 $x + x_i$ is comparable to $x + x_{i+1}$. Hence x + a is strongly connected to x + b.

Let a and b be elements of S_0. Then a and b are strongly connected to 0. By Lemma 4.12 a + b is strongly connected to a + 0 = a, so that by transitivity it is strongly connected to 0 and is therefore in S_0. Again by Lemma 4.12 -a + a = 0 is strongly connected to -a + 0 = -a whence $-a \in S_0$. Thus S_0 is a subgroup of P.

The set $a + S_0$ is contained in S_a because $a \in S_a$ and all elements of $a + S_0$ are strongly connected to a according to Lemma 4.12. Likewise, $-a + S_a$ is contained in S_0--that is, S_a is contained in $a + S_0$ and, therefore, $a + S_0 = S_a$. Similarly, $S_0 + a = S_a$ so that S_0 is a normal subgroup of P, proving the following theorem.

Theorem 4.13: S_0 is a normal subgroup of P, and the strong components of P are the cosets of S_0.

In the next section the strong components are used to characterize the order structure of dualistic poip-groups. This section is concluded with the following example illustrating the relationships among the cosets of S_0, $C_0^1(C_0)$, and C_0.

Example 4.14: Let P be the poip-group composed of the additive group of integers ordered as shown in Figure 4.15. Any block of integers in the diagram is a class of integers modulo 12. In this poip-group

$$S_0 \cup S_4 \cup S_8 = C_0^1(C_0),$$

$$S_2 \cup S_6 \cup S_{10} = C_0^4(C_0),$$

$$S_3 \cup S_7 \cup S_{11} = C_1^1(C_0),$$

$$S_1 \cup S_5 \cup S_9 = C_1^4(C_0),$$

$$C_0^1(C_0) \cup C_0^4(C_0) = C_0,$$

$$C_1^1(C_0) \cup C_1^4(C_0) = C_1.$$

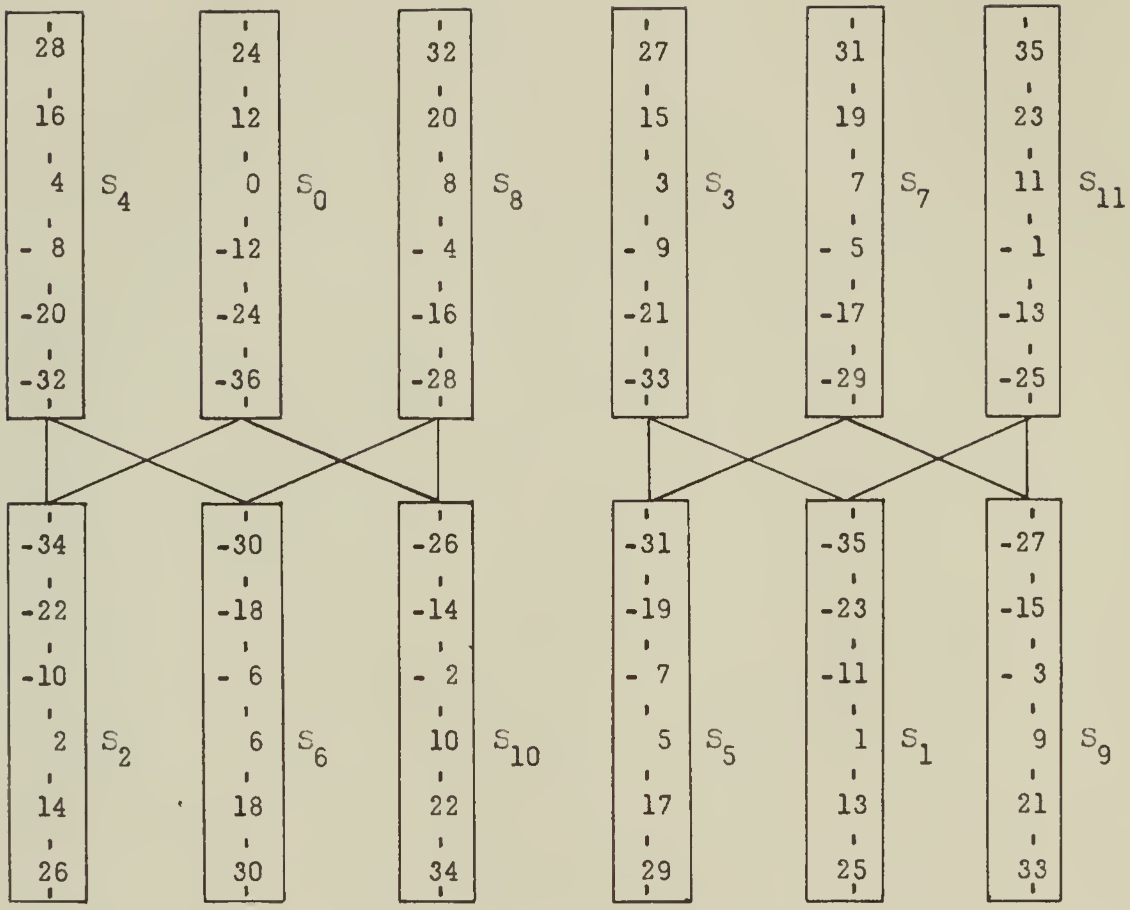

28
16
4
- 8
-20
-32
S4
24
12
0
-12
-24
-36
S0
32
20
8
- 4
-16
-28
S8
27
15
3
- 9
-21
-33
S3
31
19
7
- 5
-17
-29
S7
35
23
11
- 1
-13
-25
S11
-34
-22
-10
2
14
26
S2
-30
-18
- 6
6
18
30
S6
-26
-14
- 2
10
22
34
S10
-31
-19
- 7
5
17
29
S5
-35
-23
-11
1
13
25
S1
-27
-15
- 3
9
21
33
S9

Fig. 4.15

V. A DECOMPOSITION THEOREM

In this section it will be shown that a dualistic poip-group is order isomorphic to the ordinal product of a certain finite dimensional poip-group and a po-group.

<u>Lemma 5.1</u>: If $S_x \subset C_x^i(C_0)$ and $S_y \subset C_x^j(C_0)$, where $i \neq j$, and if $x_1 > y_1$ for some x_1 in S_x and some y_1 in S_y, then $x' > y'$ for every x' in S_x and every y' in S_y.

Consider the case in which i = 1 or 2 and j = 3 or 4. Let x_2 be a member of S_x which is comparable to x_1. If $x_2 \geq x_1$, then $x_2 > y_1$. So assume that $x_2 \leq x_1$.

<u>Case i</u>: Suppose $x_1 \in C_x^1(C_0)$. Then $-x_1$ is order preserving on the right relative to C_x, and $x_1 > y_1$ implies that $0 > y_1 - x_1$. Now $x_2 \in C_x^1(C_0)$, so that $x_2 > (y_1 - x_1) + x_2$. On the other hand, $x_1 \geq x_2$ and the fact that $-x_1$ is order preserving on the left relative to C_x imply that $0 \geq -x_1 + x_2$. Since y_1 is order reversing on the left relative to C_0, we have $y_1 \leq (y_1 - x_1) + x_2$ and, therefore, $y_1 < x_2$.

<u>Case ii</u>: Suppose $x_1 \in C_x^2(C_0)$. Then $-x_1$ is order reversing on the right relative to C_x, and $x_1 > y_1$ implies that $0 < y_1 - x_1$. Now $x_2 \in C_x^2(C_0)$, so that $x_2 > (y_1 - x_1) + x_2$. However, $x_1 \geq x_2$ and $-x_1 \in C_x^2(C_x)$ give $0 \geq -x_1 + x_2$. Now since y is order reversing on

the left relative to C_0, it is true that $y_1 \leq (y_1 - x_2) + x_2$ and hence that $y_1 < x_2$.

The above argument proves that every element of S_x which is comparable to x_1 is greater than y_1. It follows that any x' in S_x, being strongly connected to x_1, is also greater than y_1. Using a similar procedure it can be shown that any y' in S_y is less than x_1. So each $y' \in S_y$ is under x_1 and hence under every $x' \in S_x$. This establishes the cases in which $i = 1$ or 2 and $j = 3$ or 4. The dual argument establishes the remaining cases.

Definition 5.2: Order the elements of P/S_0 as follows: $S_x \geq S_y$ if and only if $S_x = S_y$ or every x' in S_x is greater than every y' in S_y. This ordering will be called the natural ordering of P/S_0.

The natural ordering of P/S_0 is obviously a partial ordering, and henceforth the symbol P/S_0 will represent the group P/S_0 ordered in this manner.

Theorem 5.3: The mapping h given by $h(x) = S_x$ is a group homomorphism of P onto P/S_0 such that $x \geq y$ implies $h(x) \geq h(y)$ and $h(x) > h(y)$ implies $x > y$. A set C is a weak component of P if and only if h(C) is a weak component of P/S_0. Furthermore, when some weak component of P/S_0 contains at least two distinct elements, every weak component of P/S_0 contains at least two elements; for any $a, x \in P$ x is of order preserving type i relative to C_a if and only if h(x) is of order preserving type i relative to $h(C_a)$ $(i = 1, 2, 3, 4)$.

Clearly h is a group homomorphism. Consider elements $x \geq y$ in $C_x = C_x^i(C_0) \cup C_x^j(C_0)$. If x and y are both elements of $C_x^i(C_0)$ or both elements of $C_x^j(C_0)$, then they are members of the same strong component S_x, so that $S_x = S_y$. However, if $x \in C_x^i(C_0)$ and $y \in C_x^j(C_0)$, where $i \neq j$, then according to the preceding lemma every element of S_x is over every element of S_y, so that $S_x > S_y$. This shows that $x \geq y$ implies $h(x) \geq h(y)$.

Suppose now that $h(x) = S_x > S_y = h(y)$. Then $S_x \neq S_y$. By Definition 5.2 every x′ in S_x is greater than every y′ in S_y, so that in particular $x > y$.

It follows from the isotone property of h (that is, the property that $x \geq y$ implies $h(x) \geq h(y)$) that if C is a weak component of P, then h(C) is a weak component of P/S_0. Conversely, since $h(x) = S_x \geq S_y = h(y)$ implies that x is connected to y, it is clear that if h(C) is a weak component of P/S_0, then C must be a weak component of P.

By Theorem 4.8 if some weak component of P/S_0 contains at least two distinct elements, then all of them contain at least two elements. Suppose $h(a) > h(b)$. When $h(x) + h(a) > h(x) + h(b)$, it is true that $h(x + a) > h(x + b)$ and $x + a > x + b$. On the other hand, $h(x) + h(a) < h(x) + h(b)$ implies $h(x + a) < h(x + b)$ and $x + a < x + b$. Thus when h(x) is of order preserving type i relative to a weak component $h(C_a)$ having at least two elements, $x \in C_x^i(C_0)$. Similarly, if $x \in C_x^i(C_a)$, then h(x) is of order preserving type i relative to $h(C_a)$.

Corollary 5.4: The system P/S_0 is a dualistic poip-group of finite dimension.

If each weak component of P/S_0 consists of merely a single element, P/S_0 is totally unordered and is obviously a dualistic poip-group. Otherwise, the result follows from Theorems 3.22 and 5.3.

Suppose that $S_x > S_y > S_z$ in P/S_0. Then by Theorem 5.3, $x > y > z$. However, x, y, and z are all in $C_x = C_x^i(C_0) \cup C_x^j(C_0)$, so that at least two of them are members of either $C_x^i(C_0)$ or $C_x^j(C_0)$. These two elements are thus strongly connected--meaning that they are in the same strong component. This is impossible since $S_x \neq S_y \neq S_z \neq S_x$. Hence there is no chain of length greater than one in P/S_0.

Corollary 5.5: Let x and y be weakly connected elements of P. If P/S_0 is of length one, x and y are of the same order preserving type relative to any given weak component of P if and only if S_x and S_y are both maximal or both minimal elements of P/S_0.

This corollary follows from the observation that when $S_x \neq S_y$, then S_x is comparable to S_y only if x and y are of different order preserving types relative to any given weak component of P and the fact that any weak component of P contains elements of only two order preserving types relative to any other weak component.

Corollary 5.6: If P/S_0 is of length one, the elements of S_x have the same order preserving properties in P as S_x has in P/S_0--

that is, x is of order preserving type i relative to C if and only if S_x is of order preserving type i relative to h(C).

Corollary 5.7: If P/S_0 is commutative and of length one, then $C_x = C_x^1(C_a) \cup C_x^4(C_a)$ for all $a, x \in P$.

Theorem 5.8: The strong component S_0 is a self-dual po-group under the relativized ordering. Furthermore, every coset of S_0 is order isomorphic to S_0, and if a is any element of the coset S of S_0, such an isomorphism may be described as follows: (i) if a is order preserving on the left relative to C_0, then $b \in S$ corresponds to $-a + b \in S_0$, and (ii) if a is order reversing on the left relative to C_0, then $b \in S$ corresponds to $-b + a \in S_0$.

Obviously S_0 is a po-set under the relativized ordering. Moreover, $S_0 \subset C_0^1(C_0)$, so that every element of S_0 is order preserving on the left and right relative to S_0. Hence S_0 is a po-group.

To show that S_0 is self-dual, let x correspond to -x for all $x \in S_0$. If $y > z$ in S_0, then $0 \geq z - y$, so that $-z \geq -y$--that is, the correspondence is an anti-isomorphism (in the order sense) and S_0 is self-dual.

Now let S be some coset of S_0 and a some element in S.

Suppose that the members of $S = a + S_0$ are order preserving on the left relative to C_0 and let $b \in S$ correspond to $x = -a + b \in S_0$ (denote this by $b \longleftrightarrow x$). The correspondence is clearly one to one. If y and z are members of S_0 such that $y \geq z$ where $c \longleftrightarrow y$

and $d \longleftrightarrow z$, then $-a + c = y \geq z = -a + d$ and, since a is order preserving on the left relative to C_0, $c \geq d$. Now if $c > d$ in S_0, where $c \longleftrightarrow y$ and $d \longleftrightarrow z$, then $y = -a + c \geq -a + d = z$ since $-a$ is order preserving on the left relative to $C_a = C_c$. Therefore, the correspondence is an order isomorphism.

If the elements of S are order reversing on the left relative to S_0, a procedure similar to that above shows that S_0 is anti-isomorphic to S under the correspondence which carries $b \in S$ into $-a + b \in S_0$. Since S_0 is anti-isomorphic to itself under the correspondence which carries every element of S_0 into its inverse, this indicates that S is order isomorphic to S_0 under the correspondence which carries $b \in S$ into $-(-a + b) = -b + a \in S_0$.

Theorem 5.9: A dualistic poip-group P is order isomorphic to the ordinal product $(P/S_0)\circ S_0$ of the finite dimensional dualistic poip-group P/S_0 and the po-group S_0. If A is a set of coset representatives, then the isomorphism may be described as follows: (i) if $b \in P$ is order preserving on the left relative to S_0, then b corresponds to $(S_b, -a + b)$ where a is in $S_b \cap A$, and (ii) if $b \in P$ is order reversing on the left relative to S_0, then b corresponds to $(S_b, -b + a)$ where a is in $S_b \cap A$.

When $b \in P$ and (S_b, y) correspond in the manner described in (i) and (ii), denote this correspondence by $b \longleftrightarrow (S_b, x)$.

Suppose $b \longleftrightarrow (S_b, y)$ and $c \longleftrightarrow (S_b, z)$. Then b and c are both members of S_b. It follows from Theorem 5.8 that $b \geq c$ if and only if $y > z$. Hence $b \geq c$ if and only if $(S_b, y) \geq (S_b, z)$ since (according to the definition of the ordinal product of two po-sets) when $u = v$, $(u, y) \geq (v, z)$ if and only if $y \geq z$.

Suppose now that $b \longleftrightarrow (S_b, y)$ and $c \longleftrightarrow (S_c, z)$, where $S_b \neq S_c$. Then $b \neq c$. If $b > c$, it is implied by Definition 5.2 that $S_b > S_c$. Hence if $b > c$, $(S_b, y) > (S_c, z)$. When $(S_b, y) > (S_c, z)$, it follows that $S_b > S_c$. Thus by Definition 5.2, $(S_b, y) > (S_c, z)$ implies that $b > c$. Therefore, $b > c$ if and only if $(S_b, y) > (S_c, z)$, completing the proof.

The above theorem is illustrated by Example 4.14 in which P/S_0 is composed of the integers modulo 12. When it is ordered in the natural manner, P/S_0 may be represented by the diagram below. The elements of the figure are, of course, added modulo 12.

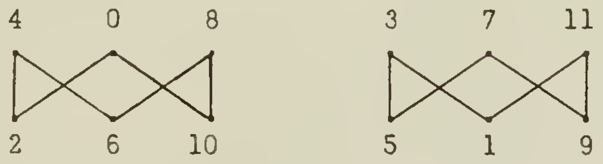


Fig. 5.10

It is easily verified that when P is of finite length, $S_0 = \{0\}$. This enables us to give the following definition.

<u>Definition 5.11</u>: A poip-group P is said to be <u>upright</u> if P/S_0 is of length one and S_0 is maximal in P/S_0. It is said to be <u>inverted</u> if P/S_0 is of length one and S_0 is minimal in P/S_0.

The dual of any theorem concerning upright poip-groups holds for inverted poip-groups.

Theorem 5.12: Let P be a dualistic poip-group, P/S_0 be of length one, and S_a be a maximal (minimal) element of P/S_0. Then x is order preserving on the left relative to C_a if and only if $x + S_a$ is maximal (minimal), and x is order reversing on the left relative to C_a if and only if $x + S_a$ is minimal (maximal).

As before, let h be the mapping defined by $h(x) = S_x$. Consider $P/S_0 = h(P)$ where $S_a = h(a)$ is maximal (minimal). Then $S_x = h(x) \in P/S_0$ is order preserving (reversing) on the left relative to $h(C_a)$ if and only if $S_{x+a} = h(x + a)$ is maximal (minimal), and it is order reversing on the left relative to $h(C_a)$ if and only if $x + a$ is minimal (maximal).

The theorem now follows from Corollary 5.6.

Corollary 5.13: Let P be an upright dualistic poip-group such that P/S_0 is of length one. An $x \in P$ is of order preserving type 1 (order preserving type 4) relative to C_0 if and only if S_x is a maximal (minimal) element of P/S_0.

Many of the results of this section have been concerned only with the case in which P/S_0 is of length one. When it is of length zero, P has a simpler form since then $S_x = C_x$ for all $x \in P$. This means that the elements of any weak component of P are all of the same order preserving type relative to any other weak component of P. Furthermore, every weak component is order isomorphic to the po-group C_0.

It was shown in this section that when P/S_0 is of length one, many of its properties are carried over to P. Also, the order structure of P can be determined if the order structure of S_0 and P/S_0 are known. Since S_0 is a po-group we need characterize only finite dimensional poip-groups.

VI. FINITE DIMENSIONAL POIP-GROUPS

The only finite dimensional poip-groups which need to be given further consideration are the dualistic poip-groups of length one. These systems are characterized below.

6.1: When a is a maximal element of a poip-group of length one, T_a will represent the set $\{x \in P \,/\, x < a\}$, and when a is minimal, it will denote the set $\{x \in P \,/\, x > a\}$.

Theorem 6.2: An upright dualistic poip-group P satisfies the following conditions: (i) if $t \in T_0$, then $-t \in T_0$; (ii) for any element s of $s' + C_0$, every element of $s' + C_0$ can be written in the form $s + t_1 + t_2 + \ldots + t_n$, $t_i \in T_0$ $(i = 1, 2, \ldots, n)$, where the elements t_i are not necessarily distinct; (iii) if $t_1 + t_2 + \ldots + t_n = 0$, $t_i \in T_0$ $(i = 1, 2, \ldots, n)$, then n is an even integer; (iv) $a + T_0 = T_0 + a$ for all $a \in P$; (v) $a + T_0 = T_a$ for all $a \in P$; and (vi) if s is maximal, then $s + t_1 + \ldots + t_n$, $t_i \in T_0$ $(i = 1, 2, \ldots, n)$, is maximal (minimal) if and only if n is even (odd). Conversely, let G be a group and C_0 a normal subgroup of G which contains a non-void subset T_0 satisfying conditions i-iv. Define an ordering on G using conditions v and vi, first assigning some representative element $s \in s' + C_0$ (let 0 be the representative element of C_0) the role of a maximal element. Under this ordering G is an upright dualistic poip-group in which C_0 is the set of all elements weakly connected to 0 and T_0 is the set of elements of G which are less than 0.

Let P be an upright dualistic poip-group.

If $t < 0$, then $-t + t = 0$ is comparable to $-t$, so that, since 0 is maximal, $-t < 0$. This establishes condition i. If $b \in T_a$, by (6.1) b is comparable to a. Hence $-a + b < 0$ and $-a + b = t$ for some $t \in T_0$. Consequently, $b = a + t$, so that b is in $a + T_0$, proving that $T_a \subset a + T_0$. Now let t be a member of T_0, whence $t < 0$. Therefore $a + t$ is comparable to a but $a + t \neq a$--that is, $a + t$ is in T_a. This implies that $T_a \supset a + T_0$ and thus $T_a = a + T_0$. Similarly, $T_a = T_0 + a$, proving conditions iv and v.

Next suppose that a is an element of C_0 not of the form $t_1 + t_2 + \ldots + t_n$, $t_i \in T_0$. There exists a sequence

$$0 = x_1, x_2, \ldots, x_n = a$$

such that x_i is comparable to x_{i+1}. Assume that k is the minimum integer such that x_k is not of the form $t_1 + t_2 + \ldots + t_n$, $t_i \in T_0$. Then $x_{k-1} = t_1 + t_2 + \ldots + t_n$ for $t_i \in T_0$ and x_{k-1} is comparable to x_k. By condition v, $x_k = x_{k-1} + t$ for some $t \in T_0$, so that $x_k = t_1 + t_2 + \ldots + t_n + t$, a contradiction. Hence every element of $s + C_0$ can be expressed as in condition ii.

Suppose $a = s + t_1 + t_2 + \ldots + t_n$ where s is maximal and $t_i \in T_0$. Now $t_1 < 0$ means $s + t_1$ is comparable to s. Since s is maximal,

$$s + t_1 < s,$$
$$s + t_1 + t_2 > s + t_1,$$
$$s + t_1 + t_2 + t_3 < s + t_1 + t_2,$$
$$s + t_1 + t_2 + t_3 + t_4 > s + t_1 + t_2 + t_3,$$

and so on. Thus $s + t_1 + t_2 + \ldots + t_n > s + t_1 + t_2 + \ldots + t_{n-1}$ if n is even and $s + t_1 + t_2 + \ldots + t_n < s + t_1 + t_2 + \ldots + t_{n-1}$ if n is odd. Hence a is maximal (minimal) if and only if n is even (odd), establishing condition vi from which condition iii follows immediately.

Conversely, let G be a group containing a normal subgroup C_0 which has a subset T_0 satisfying conditions i-iv. Suppose that $s + t_1 + t_2 + \ldots + t_n = s + t_1' + t_2' + \ldots + t_n'$ where $t_i, t_j' \in T_0$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. Then $t_1 + t_2 + \ldots + t_m - t_n' - t_{n-1}' - \ldots - t_1' = 0$. This equation, along with conditions i and iii, implies that m + n is even--that is, that either m and n are both even or both odd. Thus conditions v and vi establish a well-defined partial ordering for G. Under this ordering it is clear that G is a po-set of length one with no isolated elements.

Let $t_1 + t_2 \ldots + t_n$, $t_i \in T_0$, be an element of C_0. By (i) and (v), $t_1 + t_2 + \ldots + t_n$ is comparable to $c + t_1 + t_2 + \ldots + t_{n-1}$; $t_1 + t_2 + \ldots + t_{n-1}$ is comparable to $c + t_1 + t_2 + \ldots + t_{n-2}$, $t_1 + t_2 + \ldots + t_{n-2}$ is comparable to $c + t_1 + t_2 + \ldots + t_{n-3}$. Continuing in this manner, it follows that $t_1 + t_2 + \ldots + t_n$ is weakly connected to 0. Similarly, any element weakly connected to 0 is of the form $t_1 + t_2 + \ldots + t_n$, $t \in T_0$. Hence C_0 is a weak component of G.

Let x be some element of G. If $a > b$, then $x + a$ is comparable to $x + b$ since $a = b + t$ for some $t \in T_0$ implies $x + a = (x + b) + t$. The remainder of the proof is that of Theorem 3.9.

Example 6.3: Let G be a group which contains a commutative normal subgroup C_0 with basis elements $g_1, g_2, \ldots, g_k$ each of even order. Every element of C_0 can be expressed uniquely (apart from order) as a summation $n_1g_1 + n_2g_2 + \ldots + n_kg_k$ with $0 \leq n_i < q_i$ where q_i is the order of g_i. Let $T_0 = \{g_1, \ldots, g_k, -g_1, \ldots, -g_k\}$. Suppose that $m_1g_1 + \ldots + m_kg_k + n_1(-g_1) + \ldots + n_k(-g_k) = 0$. Then $(m_1 - n_1)g_1 + \ldots + (m_k - n_k)g_k = 0$, and $(m_i - n_i)g_i = 0$ $(i = 1, 2, \ldots, k)$. Hence $m_i - n_i = p_iq_i$ for some integer p_i, and $m_i - n_i$ is even since q_i is even. Thus $m_i + n_i$ is even, implying that $m_1 + \ldots + m_k + n_1 + \ldots + n_k = (m_1 + n_1) + \ldots + (m_k + n_k)$ is even. Therefore, T_0 satisfies conditions i-iv of Theorem 6.2.

As a particular example, consider a group G in which C_0 is generated by a and b under the conditions $a + b = b + a$, $2a = 0$, and $4b = 0$. Let $T_0 = \{a, b, 3b\}$. Then when C_0 is ordered according to conditions v and vi of Theorem 6.2, C_0 becomes the poip-group represented by the following diagram.

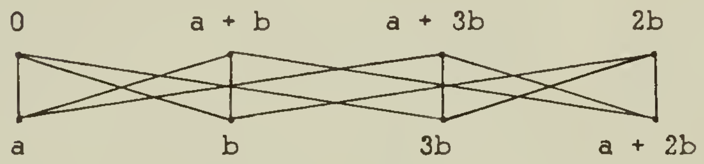


Fig. 6.4

Assign $s + a \in s + C_0$, $s \notin C_0$, the role of a maximal element of G and order $s + C_0$ according to Theorem 6.2. Then $s + C_0$ is as indicated below.

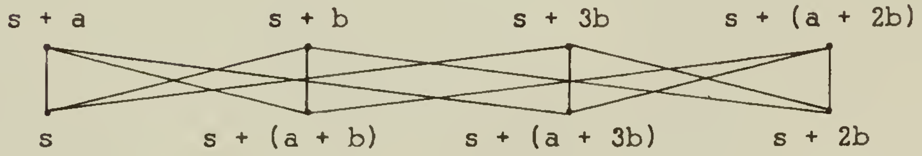


Fig. 6.5

VII. THE ALGEBRAIC CHARACTER OF POIP-GROUPS

It has been shown that many of the properties of P, in particular its structure as a po-set, can be determined when S_0 and P/S_0 are known. In this section more is said about the relationships between the latter systems and the algebraic nature of P.

Theorem 7.1: Let G be a group satisfying (i) G contains a normal subgroup S ordered so as to form a connected po-group, such that (ii) G/S is ordered so as to form an upright dualistic poip-group of length one, and (iii) $a + 0^* = 0^* + a$ for all $a \in G$. Then G can be ordered in one and only one way to form a dualistic poip-group such that the po-groups S_0 and S coincide and the poip-groups P/S_0 and G/S coincide. This ordering is described as follows: (1) if $a + S \neq b + S$, then $a > b$ if and only if $a + S > b + S$ in G/S, (2) if $a + S = b + S$ is maximal in G/S, $a \geq b$ if and only if $a - b \in 0^*$, and (3) if $a + S = b + S$ is minimal in G/S, $a \geq b$ if and only if $b - a \in 0^*$.

The ordering defined on G is obviously a reflexive and anti-symmetric relation. In order to show that it is transitive, consider $a > b$ and $b > c$ in G.

Case i: Suppose $a + S = b + S = c + S$ is maximal in G/S. Then $a - b > 0$ and $b - c > 0$. Since S is a po-group, we have $a - c = (a - b) + (b - c) > (a - b) + 0 = a - b > 0$, implying that $a - c \in 0^*$ and $a > c$.

Case ii: Suppose $a + S = b + S = c + S$ is minimal in G/S. Then $b - a > 0$ and $c - b > 0$. Therefore, $c - a = (c - b) + (b - a) > (c - b) + 0 = c - b > 0$, implying $c - a \in 0^*$ and $a > c$.

Case iii: Suppose $a + S = b + S$ is maximal and $c + S$ is minimal in G/S. Then $a + S = b + S > c + S$, proving that $a > c$.

Case iv: Suppose $a + S$ is maximal and $b + S = c + S$ is minimal in G/S. Then $a + S > b + S = c + S$, giving $a > c$.

Therefore, G is a po-set, and we want to show that under this ordering G is a dualistic poip-group.

First observe that if $x - y \in 0^*$, then $-y \in -x + 0^* = 0^* -x$, implying that $-y + x \in 0^*$.

Now consider any element $x \in G$ and $a > b$ in a weak component C. Obviously $a + S \geq b + S$.

Suppose $x + S$ is of order preserving type 1 relative to the weak component of G/S which contains $a + S$. It will be shown that x itself preserves order when added to the left and to the right of elements of C.

Case i: Suppose a and b are ordered according to (1). Then $a + S > b + S$ and $(x + a) + S = (x + S) + (a + S) > (x + S) + (b + S) = (x + b) + S$. Hence $x + a > x + b$. Similarly, $a + x > b + x$.

Case ii: Suppose a and b are ordered according to (2). This means $a - b \in 0^*$ and $a + S = b + S$ is maximal in G/S. Then $-(x + b) + (x + a) = (-b - x) + (x + a) = -b + a \in 0^*$. Therefore,

$(x + a) - (x + b) \in 0^*$. Since $x + S$ is order preserving on the left relative to the weak component of G/S containing $a + S$, and $a + S$ is maximal, then $(x + a) + S = (x + S) + (a + S)$ must be maximal. Thus $x + a > x + b$. On the other hand, $(a + x) - (b + x) = (a + x) + (-x - b) = a - b \in 0^*$. However, $(a + x) + S$ is maximal in G/S, so that $a + x > b + x$.

Case iii: Suppose a and b are ordered according to (3). The proof that $x + a > x + b$ and $a + x > b + x$ is similar to that of the preceding case.

Continuing in this manner, it can be shown that the order preserving properties of $x \in G$ are the same as those of $x + S$ in G/S. It follows from Theorem 3.22 that G under the ordering described by (1)-(3) is a dualistic poip-group P. It is easily shown that S_0 and P/S_0 coincide with S and G/S, respectively.

Now consider an upright dualistic poip-group P. Obviously it satisfies (1). If $a \geq b$, where $a + S = b + S$ is maximal in P/S_0, then $b \in C_b^1(C_0)$ and $-b \in C_{-b}^{1}(C_b)$. Hence $a - b \geq 0$ and $a - b \in 0^*$. Moreover, $a - b \in 0^*$ implies $a - b \geq 0$. Thus if $b + S$ is maximal, $a - b \in 0^*$ if and only if $a \geq b$. We have proved that P satisfies (2). Similarly, it can be shown that P satisfies (3). This establishes the uniqueness of the ordering of G described by (1)-(3).

Theorem 7.2: Let G be a group such that (i) G contains a normal subgroup S ordered so as to form a connected po-group, and (ii) for each coset S' of S either $a + 0^* = 0^* + a$ for every $a \in S'$

or $a + 0^* = 0^+ + a$ for every $a \in S'$. Then G can be ordered to form a dualistic poip-group P such that the po-groups S_0 and S coincide and P/S_0 is totally unordered. All such orderings are effected by leaving elements not in the same coset of S unordered and ordering each coset $S' \neq S$ in either of the following ways: (1) for all $a,b \in S'$, $a \geq b$ if and only if $a - b \in 0^*$, or (2) for all $a,b \in S'$, $a \geq b$ if and only if $b - a \in 0^*$.

The proof that G is a po-set under any one of these orderings is similar to that in the preceding theorem. We now show that G is a dualistic poip-group under any one of them.

Suppose $a + 0^* = 0^+ + a$ and consider $a + s$ where $s \in 0^+$. There exists $t \in G$ such that $a + s = t + a$. Then $a - s = -t + a$ and, since $-s \in 0^*$, $-t \in 0^+$. It follows that $t \in 0^*$ and $a + 0^+ \subset 0^* + a$. Likewise, $a + 0^+ \supset 0^* + a$, so that $a + 0^+ = 0^* + a$.

A similar argument will show that $a + 0^* = 0^* + a$ implies $a + 0^+ = 0^+ + a$.

Suppose $a > b$ according to (1). Then $a - b \in 0^*$. Clearly $a \in 0^* + b$ and $a + x \in 0^* + (b + x)$ for any x. This means $(a + x) - (b + x) \in 0^*$, which implies that x is either order preserving or order reversing on the right relative to any coset of S.

<u>Case i</u>: Suppose $b + 0^* = 0^* + b$. Then $a \in 0^* + b = b + 0^*$ and $x + a \in (x + b) + 0^*$. If $(x + b) + 0^* = 0^* + (x + b)$, then $(x - a) - (x + b) \in 0^*$. If $(x + b) + 0^* = 0^+ + (x + b)$, then $(x + a) - (x + b) \in 0^+$ and $(x + b) - (x + a) \in 0^*$.

Case ii: Suppose $b + 0^* = 0^+ + b$. Then $a \in 0^* + b = b + 0^+$. Clearly $x + a \in (x + b) + 0^+$. If $(x + b) + 0^+ = 0^* + (x + b)$, then $(x + a) - (x + b) \in 0^*$. If $(x + b) + 0^+ = 0^+ + (x + b)$, then $(x + a) - (x + b) \in 0^+$ whence $(x + b) - (x + a) \in 0^*$.

The proofs of the above cases imply that x is either order preserving or order reversing on the left relative to any coset of S. The case in which $a > b$ according to (2) can be proved in a similar manner. By Theorem 3.22, G is a dualistic poip-group under any one of the orderings described in (1) and (2).

We now state an important corollary of the above theorems.

Corollary 7.3: Necessary and sufficient conditions that a group G can be ordered to form an infinite dimensional poip-group are: (1) G contains a normal subgroup $S \neq \{0\}$ which can be ordered so that it forms a weakly connected po-group, such that (ii) for each coset $S' \neq S$, either $a + 0^* = 0^* + a$ for every $a \in S'$ or $a + 0^* = 0^+ + a$ for every $a \in S'$.

The sufficiency of the conditions has been shown. Let P be an infinite dimensional poip-group. To establish the necessity we need show only that P satisfies condition ii, since it obviously satisfies (i).

Suppose that $a \in C_a^2(C_0)$. If $b \in a^*$, then $b \geq a$, so that $-a + b \geq 0$ and $b - a \leq 0$. Hence $b \in a + 0^*$ and $b \in 0^+ + a$, implying that $a^* \subset a + 0^*$ and $a^* \subset 0^+ + a$. If $c \in 0^*$ and $d \in 0^+$, then $c \geq 0$ and $d \leq 0$, so that $a + c \geq a$ and $d + a \geq a$. Thus $a + c \in a^*$ and $d + a \in a^*$, giving $a^* = a + 0^*$ and $a^* = 0^+ + a$.

Similarly, $a \in C_a^1(C_0)$, $a \in C_a^3(C_0)$, and $a \in C_a^4(C_0)$ imply $a + 0^* = a^* = 0^* + a$, $a + 0^* = a^+ = 0^+ + a$, and $a + 0^* = a^+ = 0^* + a$, respectively. Since $S_a \subset C_a^i(C_0)$ for some i, the result follows.

The following theorem occurs in (4, p. 214).

Theorem 7.4: Any po-group S is determined to within isomorphism by the set $R = 0^*$, since $a \geq b$, $a - b \in R$, and $-b + a \in R$ are equivalent conditions. Moreover, (i) $0 \in R$, (ii) if $a,b \in R$, then $a + b \in R$, (iii) if $a,b \in R$ and $a + b = 0$, then $a = b = 0$, (iv) for all $a \in S$, $a + R = R + a$. Conversely, if S is any group, and R is a subset of S satisfying (i)-(iv), then S can be ordered to form a po-group by defining $a \geq b$ in S to mean $a - b \in R$.

It is obvious that a po-group S is weakly connected if and only if, for every pair $a,b \in S$, there exists a sequence

$$a = x_1, x_2, \ldots, x_n = b$$

of elements of S such that either $x_i - x_{i+1} \in R$ or $x_{i+1} - x_i \in R$ $(i = 1, 2, \ldots, n - 1)$.

The previous corollary can now be stated in purely algebraic terms.

Theorem 7.5: Necessary and sufficient conditions that a group G can be ordered to form an infinite dimensional dualistic poip-group are: (i) G has a normal subgroup $S \neq \{0\}$ which contains a subset R satisfying (ii) $0 \in R$; (iii) if $a,b \in R$, then $a + b \in R$; (iv) if $a,b \in R$ and $a + b = 0$, then $a = b = 0$; (v) for all $a \in S$,

$a + R = R + a$; (vi) for every pair $a, b \in S$, there exists a sequence

$$a = x_1, x_2, \ldots, x_n = b$$

of elements of S such that either $x_i - x_{i+1} \in R$ or $x_{i+1} - x_i \in R$ $(i = 1, 2, \ldots, n - 1)$; and (vii) for each coset $S' \neq S$, either $a + R = R + a$ for all $a \in S'$ or $a + R = -R + a$ for all $a \in S'$.

Theorems 7.1 and 7.2 also lead to methods for constructing infinite dimensional poip-groups from a given dualistic poip-group of finite dimension and a given weakly connected po-group. These are described below.

In the discussion of the direct sum of two groups A and B, the symbol (a, B'), where $a \in A$ and B' is a subset of B, represents the set of all pairs (a, b') such that $b' \in B'$. The symbol (A', b), where A' is a subset of A and $b \in B$, is defined accordingly.

Theorem 7.6: Let P be the direct sum $G \dot{+} S$ of an upright dualistic poip-group G and a weakly connected po-group S. Then P is a dualistic poip-group when it is ordered as follows: (i) if $g_1 \neq g_2$, $(g_1, s_1) > (g_2, s_2)$ if and only if $g_1 > g_2$ in G; (ii) if $g_1 = g_2$ is maximal, $(g_1, s_1) > (g_2, s_2)$ if and only if $s_1 > s_2$ in S; and (iii) if $g_1 = g_2$ is minimal, $(g_1, s_1) > (g_2, s_2)$ if and only if $s_1 < s_2$ in S. Furthermore, S_0 coincides with $(0, S)$, and P/S_0 coincides with $(G, 0)$.

The theorem follows directly from Theorem 7.1 once we have shown that $(g, s) + (0, 0)^* = (0, 0)^* + (g, s)$ for every element

$(g, s) \in G \dotplus S$. Here it is understood that $(0, S)$ is ordered such that $(0, s_1) \geq (0, s_2)$ if and only if $s_1 \geq s_2$, and that $(G, 0)$ is ordered such that $(g_1, 0) \geq (g_2, 0)$ if and only if $g_1 \geq g_2$. But $(g, s) + (0, 0)^* = (g, s) + (0, 0^*) = (g, s + 0^*) = (g, 0^* + s) = (0, 0^*) + (g, s) = (0, 0)^* + (g, s)$.

Theorem 7.7: Let P be the direct sum $G \dotplus S$ of a group G and a weakly connected po-group S. Then P is a dualistic poip-group when it is ordered in any one of the ways indicated below:
(i) $g_1 \neq g_2$ implies (g_1, s_1) and (g_2, s_2) are not comparable;
(ii) $(0, s_1) > (0, s_2)$ if and only if $s_1 > s_2$; $g \neq 0$ implies either
(iii) for all $s_1, s_2 \in S$, $(g, s_1) > (g, s_2)$ if and only if $s_1 > s_2$
or (iv) for all $s_1, s_2 \in S$, $(g, s_1) > (g, s_2)$ if and only if $s_1 < s_2$. Moreover, for any one of these orderings, S_0 coincides with $(0, S)$ and $P/S_0 = (G, 0)$ is totally unordered.

The proof of the theorem is essentially that of Theorem 7.6.

Example 7.8: Let G be the poip-group consisting of the group I_2 ordered by setting $0 > 1$, and let S be the po-group consisting of the even integers in their natural order. When $G + S$ is ordered according to Theorem 7.6, we have

$$. . . > (0, 2) > (0, 0) > (0, -2) > . . . >$$
$$(1, -2) > (1, 0) > (1, 2) > . . .$$

Comparing this example to the poip-group P of Example 2.19 which is not the direct sum of P/S_0 and S_0, even though P/S_0 and S_0

are order isomorphic and group isomorphic (under the same correspondence) to G and S, respectively, shows that the algebraic structure of a poip-group P is not completely determined by the systems S_0 and P/S_0.

In this section we have given necessary and sufficient conditions that a group can be ordered to form an infinite dimensional poip-group. In addition, we have exhibited the relationships between the algebraic properties of a dualistic poip-group P and the structures of S_0 and P/S_0, and have shown that the latter two systems do not always completely determine P as a group.

VIII. LOIP-GROUPS

The preceding results are much simpler to state if it is assumed that the poip-group itself is weakly connected.

A weakly connected dualistic poip-group P is a po-group if and only if P/S_0 is of length zero--that is, if and only if $P = S_0$. Otherwise, P is the union of C_0^1 and its coset C_0^4. Moreover, it can easily be shown that, under the relativized ordering, C_0^1 is a po-group and C_0^4 is order isomorphic to C_0^1 under the correspondence carrying $a \in C_0^1$ into $-a + r \in C_0^4$, where r is the coset representative of C_0^4. As a consequence of Corollary 5.5, no member of C_0^1 is less than an element of C_0^4 when P is upright. These results are summarized in the following theorem.

<u>Theorem</u> <u>8.1</u>: If P is an upright dualistic poip-group which is weakly connected, then (i) $P = C_0^1 \cup C_0^4$, where (ii) C_0^1 is a normal subgroup of P with cosets C_0^1 and C_0^4, (iii) under the relativized ordering C_0^4 is order isomorphic to the po-group C_0^1, and (iv) no element of C_0^1 is under an element of C_0^4.

When every pair of elements of P is bounded in some manner, conditions i-iv are strengthened as shown in the corollaries below.

Corollary 8.2: If P is an upright dualistic poip-group in which every pair of elements has either an upper or a lower bound, it satisfies conditions i-iii, and (iv)' every element of C_0^1 is greater than every element of C_0^4.

Clearly P satisfies conditions i-iv. Now let a and b be members of C_0^1 and C_0^4, respectively, and suppose that they have an upper bound c. By condition iv, $c \notin C_0^4$, so that $c \in C_0^1$ and $c \in S_a$. However, $c > b$ and $S_a = S_c > S_b$ whence $a > b$. Similarly, if a and b have a lower bound, $a > b$.

Corollary 8.3: If P is an upright dualistic poip-group which is either up-directed or down-directed, it satisfies (i), (ii), (iii), (iv)', and (v) C_0^1 and C_0^4 are strong components of P.

Corollary 8.4: If P is an upright loip-group, it satisfies (i), (ii), (iv)', (v), and (iii)' under the relativized ordering C_0^1 is an l-group and C_0^4 is order isomorphic to it.

By condition iii, C_0^1 is a po-group under the relativized ordering. If a and b are elements of C_0^1, by (iv) $a \cup b \in C_0^1$. Now assume $a \cap b \in C_0^4$, so that every member of C_0^4 must be less than $a \cap b$, in which case $a \cap b$ is a maximal element of C_0^4. Condition iii thus implies that C_0^1 has a maximal element. However, the only l-group with a maximal element is the group of order one. Thus $a \cap b \in C_0^1$, and C_0^1 is a loip-group.

As a result of this corollary many of the results concerning l-groups are easily extended to loip-groups. We note several examples. In an l-group L, $x + (a \cup b) = (x + a) \cup (x + b)$ for all $a,b,x \in L$. In a loip-group P, this becomes $x + (a \cup b) = (x + a) \cup (x + b)$ if $x \in C_0^1$ and $x + (a \cup b) = (x + a) \cap (x + b)$ if $x \in C_0^4$. Every l-group and every loip-group is a distributive lattice. In (4, p. 222) it is shown that the congruence relations on an l-group L are the partitions of L into the cosets of its different l-ideals. On the other hand, the congruence relations on a loip-group P are the partitions of P into the cosets of its k-ideals, where the k-ideals are P and its normal subgroups which are l-ideals of C_0^1. These results are all easily established.

IX. SUMMARY

The order structure of partially ordered ideal preserving groups was determined in terms of partially ordered groups; many results concerning these latter systems can easily be extended to the more general partially ordered ideal preserving groups. Necessary and sufficient conditions that a group can be ordered to form a non-trivial partially ordered ideal preserving group were obtained. The principal concepts employed in this dissertation are the equivalence relations called weak and strong connectivity.

The investigation of partially ordered ideal preserving rings and fields is suggested as a problem for further research.

LITERATURE CITED

1. O. Frink, Ideals in partially ordered sets, Amer. Math. Monthly. vol. 61 (1954) pp. 223-34.

2. A. T. Butson, "Simply ordered ideal preserving groups." Paper read before the meeting of the American Mathematical Society, Rochester, N. Y., December, 1956.

3. C. C. MacDuffee, An introduction to abstract algebra. New York, J. Wiley and Sons, 1940.

4. G. Birkhoff, Lattice theory. American Mathematical Society Colloquium Publications, vol. XXV. New York, American Mathematical Society, 1948.

BIOGRAPHICAL SKETCH

Jan Frederick Andrus was born September 17, 1932, in Washington, D. C. He was graduated from the College of Charleston in Charleston, South Carolina, in May, 1954, with the degree Bachelor of Science. Work for the degree Master of Arts was undertaken at Emory University, Emory University, Georgia, and the degree was granted in August, 1955. The author's undergraduate major was in chemistry with a minor in mathematics; his graduate major has been mathematics, and his minor work has been in the field of physics.

During his studies the author has held assistantships in chemistry at Emory University and in mathematics at the University of Florida. He was employed in the summer of 1957 as an aeronautical research engineer with the National Advisory Committee for Aeronautics at Langley Field, Virginia.

While pursuing undergraduate study, the author was elected to membership in Sigma Alpha Phi, College of Charleston honorary society. He was awarded the S. Keith Johnson medal for outstanding achievement in science and a similar award for the highest achievement in mathematics. He was elected to membership in Alpha Sigma Phi, national physics honorary, while a graduate student at Emory University.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Date June 9, 1958

C. F. Byers
Dean, College of Arts and Sciences

Dean, Graduate School

SUPERVISORY COMMITTEE:

E. H. Hadlock
Chairman

Alton T. Butson
Co-Chairman

J. O. Moore

R. C. Meacham

C. B. Smith

W. R. Hutcherson

Donald L. Lafferty